

Revisit Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint

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Abstract—In this paper, we revisit the topic "Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint". We also revisit the topic "New Approach to Synchronization Analysis of Linearly Coupled Ordinary Differential Systems" [2], [3]. It is revealed that two topics are closely relating to each other. The relationship between [1] and [2] as well as the relationship between consensus and synchronization is revealed.

Index Terms—Consensus, synchronization,.

I. INTRODUCTION

In recent decades, the synchronization problem of multiagent systems has received compelling attention from various scientific communities due to its broad applications. Many natural and synthetic systems, such as neural systems, social systems, WWW, food webs, electrical power grids, can all be described by complex networks. In such a network, every node represents an individual element of the system, while edges represent relations between nodes. For decades, complex networks have been focused on by scientists from various fields, for instance, sociology, biology, mathematics and physics.

In the pioneer work [5] (also see [6]), the authors proposed a master stability function near a trajectory, by which local synchronization was investigated. In [7], a distance between node state and synchronization manifold was introduced and global synchronization was discussed.

In [2], a general framework is presented to analyze synchronization stability of Linearly Coupled Ordinary Differential Equations (LCODEs). The uncoupled dynamical behavior at each node is general, which can be chaotic or others; the coupling configuration is also general, without assuming the coupling matrix be symmetric or irreducible. It was revealed that the left and right eigenvectors corresponding to eigenvalue zero of the coupling matrix play key roles in the stability analysis of the synchronization manifold. Different from previous papers, a non-orthogonal projection on the synchronization manifold was first introduced. With this projection, a new approach to investigate the stability of the synchronization manifold of coupled oscillators was proposed. Novel master stability function near the projection was proposed.

It is clear that linearly Coupled linear system as well as consensus are special cases of the linearly Coupled Ordinary

Differential Equations (LCODEs), which are also hot topics in decades. For example, the synchronization of observer based linear systems (see [8], [9], [1] and others).

II. UNIFIED MODEL AND GENERAL APPROACH

In this section, we present some definitions, denotations and lemmas required throughout the paper.

In [2], following model was discussed

$$\frac{dx^i(t)}{dt} = f(x^i(t), t) + \sum_{j=1}^N l_{ij} \Gamma x^j(t), \quad i = 1, \dots, N \quad (1)$$

where $x^i(t) \in R^n$ is the state variable of the i -th node, $t \in [0, +\infty)$ is a continuous time, $f : R \times [0, +\infty) \rightarrow R^n$ is continuous map, $L = (l_{ij}) \in R^{N \times N}$ is the coupling matrix with zero-sum rows and $l_{ij} \geq 0$, for $i \neq j$, which is determined by the topological structure of the LCODEs, and $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ with $\gamma_i \geq 0$, for $i = 1, \dots, n$.

Later, following observer based synchronization

$$\frac{dx^i(t)}{dt} = f(x^i(t), t) + \sum_{j=1}^N l_{ij} F C x^j(t), \quad i = 1, \dots, N \quad (2)$$

where $y(t) = Cx(t)$ is observer measurement $C \in R^{q \times n}$, and $F \in R^{n \times q}$ was discussed in [1].

In [8], [9],

$$\frac{dx^i(t)}{dt} = A x^i(t) + B \tilde{L} \sum_{j=1}^N c l_{ij} x^j(t), \quad i = 1, \dots, N, \quad (3)$$

where $\tilde{L} \in R^{p \times n}$, was discussed. It is a special case when the relative states between neighboring agents are available.

As a very special case, when $f = 0$ and $n = 1$, it is just consensus problem

$$\frac{dx^i(t)}{dt} = \sum_{j=1}^N l_{ij} x^j(t), \quad i = 1, \dots, N, \quad (4)$$

Following Lemma can be found in [2] (see Lemma 1 in [2]).

Lemma 1. *If L is a coupling matrix with $\text{Rank}(L) = N-1$, then the following items are valid:*

- 1) *If λ is an eigenvalue of L and $\lambda \neq 0$, then $\text{Re}(\lambda) < 0$;*
- 2) *L has an eigenvalue 0 with multiplicity 1 and the right eigenvector $[1, 1, \dots, 1]^T$;*

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- 3) Suppose $\xi = [\xi_1, \xi_2, \dots, \xi_m]^\top \in R^m$ (without loss of generality, assume $\sum_{i=1}^m \xi_i = 1$) is the left eigenvector of A corresponding to eigenvalue 0. Then, $\xi_i \geq 0$ holds for all $i = 1, \dots, m$; more precisely,
- 4) L is irreducible if and only if $\xi_i > 0$ holds for all $i = 1, \dots, m$;
- 5) L is reducible if and only if for some i , $\xi_i = 0$. In such case, by suitable rearrangement, assume that $\xi^\top = [\xi_+^\top, \xi_0^\top]$, where $\xi_+ = [\xi_1, \xi_2, \dots, \xi_p]^\top \in R^p$, with all $\xi_i > 0$, $i = 1, \dots, p$, and $\xi_0 = [\xi_{p+1}, \xi_{p+2}, \dots, \xi_N]^\top \in R^{N-p}$ with all $\xi_j = 0$, $p+1 \leq j \leq N$. Then, L can be rewritten as $\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$ where $L_{11} \in R^{p \times p}$ is irreducible and $L_{12} = 0$.

By definition, any reducible coupling matrix can be rewritten as (see [3])

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$$

or more generally, (see [2])

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{p1} & L_{p2} & \cdots & L_{pp} \end{bmatrix}$$

where and for each $q = 2, \dots, p$, $L_{qq} \in R^{m_q \times m_q}$, is irreducible.

Remark 1. (see [2]) In fact, a coupling matrix $L = [l_{ij}] \in R^{N \times N}$ is a singular M-matrix. Thus,

- If λ is an eigenvalue of L then $\text{Re}(\lambda) \leq 0$;
- L has a spanning tree with root L_{11} , if and only if for any $q = 2, \dots, p$, there is some $\bar{q} < q$, such that $L_{\bar{q}q} \neq 0$;
- By M-matrix theory, for any $q = 2, \dots, p$, L_{qq} is a non-singular matrix, if and only if there is some $\bar{q} < q$, such that $L_{\bar{q}q} \neq 0$. Equivalently, L_{qq} is a singular matrix (a coupling matrix), if and only if for all $\bar{q} < q$, $L_{\bar{q}q} = 0$. In this case, L_{11} is not a root and L has no spanning tree;
- Therefore, L has an eigenvalue 0 with multiplicity 1, if and only if L has a spanning tree with root L_{11} .

It is also clear that the so-called master-slave system is a special case of this model. The nodes in root are masters and others are slaves.

Let $[\xi_1, \dots, \xi_N]^\top$ be the left eigen-vector corresponding to the eigenvalue 0 for the matrix $L = [l_{ij}]$. For the model (5) with directed coupling, a nonorthogonal projection of $x(t)$ on the synchronization manifold S , $\bar{X}(t) = [\bar{x}(t), \dots, \bar{x}(t)]^\top$, where $\bar{x}(t) = \sum_{i=1}^N \xi_i x_i(t)$, was first introduced in [2]. It plays a key role in discussing synchronization problem. For the orthogonal projection $\bar{x}(t) = \frac{1}{N} \sum_{i=1}^N x_i(t)$ see [3]. Based on the projection, synchronization is reduced to proving the distance between all nodes $x_i(t)$ and the synchronization manifold $\delta x_i(t) = x_i(t) - \bar{x}(t) \rightarrow 0$. And (1) can be rewritten as

$$\frac{d\delta x^i(t)}{dt} = Df(\bar{x}(t), t)\delta x^i(t) + \sum_{j=1}^m l_{ij}\Gamma\delta x^j(t) \quad (5)$$

Following theorem was proved in [2] (see Theorem 1 in [2]).

Theorem 1. Let $\lambda_2, \lambda_3, \dots, \lambda_l$ be the non-zero eigenvalues of the coupling matrix L . If all variational equations

$$\frac{dz(t)}{dt} = [Df(\bar{x}(t), t) + \lambda_k \Gamma]z(t), \quad k = 2, 3, \dots, l \quad (6)$$

are exponentially stable, then the synchronization manifold S is local exponentially stable for the general synchronization model (1).

Remark 2. The inner coupling matrix is assumed as a diagonal matrix. It can be any inner coupling matrix.

Based on stabilizable and detectable theory for linear systems, in [1], authors discussed consensus of multiagent systems and synchronization of complex networks with a unified viewpoint.

Given a linear system

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad y_i(t) = Cx_i(t) \quad (7)$$

where $x_i(t) \in R^n$ is the stat, $u_i(t) \in R^p$ is the control input, and $y_i(t) \in R^q$ is the measured output. $A \in R^{n \times n}$, $B \in R^{n \times p}$, $C \in R^{q \times n}$. It is assumed that is stabilizable and detectable.

The purpose of [1] is to discuss consensus of the linear system. As said in [1], there are two fundamental questions about the consensus problem of multiagent systems: how to reach consensus and consensus on what.

Instead of coupling $x_i(t)$ (because they are not available), the authors coupled the measured output

$$\dot{\zeta}_i(t) = \sum_{j=1}^N l_{ij}y_j(t) \quad (8)$$

where $L = [l_{ij}] \in R^{N \times N}$ is the coupling matrix among different nodes satisfying $l_{ij} \geq 0, i \neq j, l_{ii} = -\sum_{j=1, j \neq i}^N l_{ij}, i = 1, \dots, N$.

An observer-type consensus protocol

$$\dot{v}_i(t) = (A + BK)v_i(t) + F\left(\sum_{j=1}^N Cl_{ij}v_j(t) - \zeta_i(t)\right) \quad (9)$$

is proposed, which can also be written as

$$\begin{cases} \dot{v}_i(t) = (A + BK)v_i(t) + \sum_{j=1}^N FCl_{ij}(v_j(t) - x_j(t)) \\ \dot{x}_i(t) = Ax_i(t) + BKv_i(t) \end{cases} \quad (10)$$

where $K \in R^{p \times n}$, $F \in R^{n \times q}$.

The main theoretical result reported in [1] is the following *Theorem 1 ([1])* For a directed network of agents with communication topology \mathcal{G} that has a directed spanning tree, protocol (3) solves the consensus problem if and only if all matrices $A + BK$, $A + c\lambda_i FC$, $i = 2, \dots, N$, where λ_i are Hurwitz, where $i = 2, \dots, N$, are the nonzero eigenvalues of the Laplacian matrix \mathcal{L} of \mathcal{G} .

In this paper, we revisit this issue and some closely relating topics and reveal the relationship among these papers. It will be shown that unified model and more general results have

been given in [2], [3], and the results obtained in [1] can be easily derived by those given in [2], [3].

Let $e_i(t) = v_i(t) - x_i(t)$, one can transfer (10) to

$$\dot{e}_i(t) = Ae_i(t) + FC \sum_{j=1}^N l_{ij} e_j(t) \quad (11)$$

and

$$\dot{x}_i(t) = (A + BK)x_i(t) + BK e_i(t) \quad (12)$$

Denote $\bar{x}(t) = \sum_{i=1}^N \xi_i x_i(t)$, $\bar{e}(t) = \sum_{i=1}^N \xi_i e_i(t)$, $\delta x_i(t) = e_i(t) - \bar{x}(t)$, $\delta \bar{e}_i(t) = e_i(t) - \bar{e}(t)$. Then, (11) and (12) can be rewritten as

$$\frac{d}{dt} \delta \bar{e}_i(t) = A \delta \bar{e}_i(t) + \sum_{j=1}^N a_{ij} FC \delta \bar{e}_j(t) \quad (13)$$

and

$$\frac{d}{dt} \delta x_i(t) = (A + BK) \delta x_i(t) + BK \delta e_i(t) \quad (14)$$

with

$$\dot{\bar{e}}(t) = A \bar{e}(t) \quad (15)$$

It is claimed in [1] that Similar to [23], δ is referred to as the disagreement vector. In fact, the accurate saying is that it came from [3] and [2]. In particular, from [2].

Remark 3. It is clear that let $f(x^i(t), t) = Ae_i(t)$, $\Gamma = FC$. general consensus model (11) is a special case of (1). And all the results for model (1) can apply to (11).

Noticing that for linear system, local stability is equivalent to global stability. Therefore, as a special case of Theorem 1, we have

Theorem 2. Let $\lambda_2, \lambda_3, \dots, \lambda_l$ be the non-zero eigenvalues of the coupling matrix L . If

$$\frac{dz(t)}{dt} = [A + \lambda_k FC]z(t), \quad k = 2, 3, \dots, l \quad (16)$$

are exponentially stable, then $\delta \bar{e}_i(t) = e_i(t) - \bar{e}(t)$, $i = 1, \dots, N$, converge to zero exponentially.

Lemma 1. If

$$\dot{x}(t) = Ax(t) + z(t) \quad (17)$$

A is Hurwitz and $z(t) = O(e^{-\alpha t})$ then $x(t) \rightarrow 0$ exponentially.

By Theorem 2, $\delta \bar{e}_i(t) \rightarrow 0$ exponentially, $i = 1, \dots, N$, and $e_i(t) = v_i(t) - x_i(t)$ reaches synchronization exponentially.

In addition to $\delta \bar{e}_i(t) \rightarrow 0$ exponentially, if $A + BK$ is Hurwitz, by (14) and Lemma 1, we have $\delta x_i(t)$ converges to zero exponentially for all $i, j = 1, \dots, N$.

Furthermore, by

$$\dot{v}_i(t) = (A + BK)v_i(t) + FC \sum_{j=1}^N l_{ij} \delta e_j(t) \quad (18)$$

we have $v_i(t) \rightarrow 0$ exponentially.

By (15), we have

$$\bar{e}(t) = e^{At} \bar{e}(0) \quad (19)$$

and for all $i = 1, \dots, N$,

$$x_i(t) - v_i(t) - e^{At} \bar{e}(0) \rightarrow 0 \quad (20)$$

which implies (for $v_i(t) \rightarrow 0$)

$$x_i(t) - e^{At} \bar{e}(0) = x_i(t) - e^{At} \sum_{i=1}^N \xi_i [x_i(0) - v_i(0)] \rightarrow 0 \quad (21)$$

and the claim

$$x_i(t) - e^{At} \bar{e}(0) = x_i(t) - e^{At} \sum_{i=1}^N \xi_i (x_i(0)) \rightarrow 0 \quad (22)$$

holds only when $v_i(0) = 0$ for all $i = 1, \dots, m$.

Remark 4. In [1], it was said that [2], [5] (References [22] and [27] in [1]) addressed the synchronization stability of a network of oscillators by using the master stability function method.

The authors also said that the proposed framework is, in essence, consistent with the master stability function method used in the synchronization of complex networks and yet presents a unified viewpoint to both the consensus of multi-agent systems and the synchronization of complex networks.

In fact, for linear systems, global stability and local stability are equivalent. Therefore, the master stability function method can be used to prove local stability as well as global stability. It should be pointed out that the master stability functions are different in the two papers [2] and [5]. In [2], master stability function applies based on $\bar{x}(t)$. Instead, in [5], master stability function applies based on $s(t)$ satisfying $\dot{s}(t) = f(s(t))$. Here, in [1], the authors follow the line and approach proposed in [2].

Remark 5. As said in [1], there are two fundamental questions about the consensus problem of multiagent systems: how to reach consensus and consensus on what.

In fact, Theorem 1 provides a general approach to answer the question "how to reach consensus" and $\bar{x}(t)$ gives the solution of consensus (synchronization).

In [1], the so called relative-State consensus protocol (also see [8], [9])

$$\frac{dx^i(t)}{dt} = Ax^i(t) + B\tilde{L} \sum_{j=1}^N cL_{ij} x^j(t), \quad i = 1, \dots, N, \quad (23)$$

where $\tilde{L} \in R^{p \times n}$, was discussed. As a direct consequence of Theorem 1 given in [2], we have

Theorem 3. Let $\lambda_2, \lambda_3, \dots, \lambda_l$ be the non-zero eigenvalues of the coupling matrix L . If

$$\frac{dz(t)}{dt} = [A + \lambda_k B\tilde{L}]z(t), \quad k = 2, 3, \dots, l \quad (24)$$

are exponentially stable, then the synchronization manifold S is exponentially stable for the general model (23).

On the other hand, in case that the relative states between neighboring agents are not available, following protocol

$$\frac{dx^i(t)}{dt} = Ax^i(t) + B\tilde{L} \sum_{j=1}^N cL_{ij} y^j(t), \quad i = 1, \dots, N, \quad (25)$$

where $\bar{L} \in R^{p \times q}$, $B \in R^{n \times p}$, can be used.

It can also be rewritten as

$$\frac{dx^i(t)}{dt} = Ax^i(t) + B\bar{L}C \sum_{j=1}^N cL_{ij}x^j(t), \quad i = 1, \dots, N, \quad (26)$$

Therefore, by Theorem 1, we have

Theorem 4. *Let $\lambda_2, \lambda_3, \dots, \lambda_l$ be the non-zero eigenvalues of the coupling matrix L . If*

$$\frac{dz(t)}{dt} = [A + \lambda_k B\bar{L}C]z(t), \quad k = 2, 3, \dots, l \quad (27)$$

are exponentially stable, then the synchronization manifold S is exponentially stable for the general model (25).

Corollary 1, Corollary 2 in [1] can be obtained directly from Theorem 1, which has been given in [2].

In [1], it is claimed that "It is observed by comparing Theorem 2 and Corollary 2 that even if the consensus protocol takes the dynamic form (3) or the static form (22), the final consensus value reached by the agents will be the same, which relies only on the communication topology, the initial states, and the agent dynamics."

However, for the coupled system (4), we have

$$x_i(t) - e^{At} \sum_{i=1}^N \xi_i [x_i(0) - v_i(0)] \rightarrow 0 \quad (28)$$

exponentially. Instead, for the system (16), we have

$$x_i(t) - e^{At} \sum_{i=1}^N \xi_i x_i(0) \rightarrow 0 \quad (29)$$

They are different.

Remark 6. *As pointed out that*

$$x_i(t) - e^{At} \sum_{i=1}^m \xi_i [x_i(0) - v_i(0)] \rightarrow 0 \quad (30)$$

which depends heavily on the matrix A .

If A has some eigenvalues with positive real part, then all $x_i(t) \rightarrow \infty$, $i = 1, \dots, N$. Instead, if all eigenvalues of A are with negative real part, then all $x_i(t) \rightarrow 0$, $i = 1, \dots, N$.

It is clear that only the case when all eigenvalues of A are with negative real part except some are with real part 0 is meaningful.

It has also been addressed in [1] (Remark 3) that the matrix A in ((1) in [1]) having eigenvalues along the imaginary axis is critical for the agents to reach consensus at a nonzero value under protocol ((3) in [1]). Typical examples of this case include the single and double integrators considered in the existing literature [10]-[13] ([23],[29], [31], [32] in [1]). It should be pointed out that similar results have been obtained for some special cases of this theorem, e.g., in [32] where the agent dynamics are assumed to be double integrators and in [39] where the consensus protocol is static.

In summary, the model

$$\frac{dx^i(t)}{dt} = f(x^i(t), t) + \sum_{j=1}^N a_{ij} \Gamma x^j(t), \quad i = 1, \dots, N \quad (31)$$

was discussed and investigated in [2]. Non-orthogonal projection was introduced and Theorem 1 was proved.

In [1], following model

$$\begin{cases} \dot{v}_i(t) = (A + BK)v_i(t) + \sum_{j=1}^N FCa_{ij}(v_j(t) - x_j(t)) \\ \dot{x}_i(t) = Ax_i(t) + BKv_i(t) \end{cases} \quad (32)$$

was proposed and Theorem 2 was proved.

It is pointed out that Theorem 2 can be derived easily from Theorem 1.

Moreover, the following model

$$\frac{dx^i(t)}{dt} = Ax^i(t) + B\bar{L} \sum_{j=1}^N cL_{ij}x^j(t), \quad i = 1, \dots, N, \quad (33)$$

is a special case of the model (1), just letting $f(x(t)) = Ax(t)$ and $\Gamma = B\bar{L}$.

It is clear that the consensus model

$$\frac{dx^i(t)}{dt} = \sum_{j=1}^N L_{ij}x^j(t), \quad i = 1, \dots, N, \quad (34)$$

discussed in [10] is a special case of the synchronization model, just letting $f(x(t)) = 0$, $n = 1$ and $\Gamma = 1$.

In the same time, [3] analyses synchronization of linearly coupled map lattices (LCMLs)

$$x^i(t+1) = f(x^i(t)) + \sum_{j=1}^N L_{ij}f(x^j(t)), \quad i = 1, \dots, N, \quad (35)$$

Let $f(x) = x$, then it is the discrete consensus model

$$x^i(t+1) = x^i(t) + \sum_{j=1}^N L_{ij}x^j(t), \quad i = 1, \dots, N, \quad (36)$$

In [3], the mean $\bar{x}(t) = \frac{1}{m}x_i(t)$ was introduced. And the approach to synchronization by proving $\|x_i(t) - \bar{x}(t)\| \rightarrow 0$ was addressed.

As for continuous version, see [2], [4] or early pioneer paper [5].

Therefore, many results on synchronization of coupled linear systems or consensus can be obtained as special cases of the general model on synchronization of coupled nonlinear systems (1).

III. SECOND ORDER CONSENSUS

In [11], following second order consensus model was proposed

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \sum_{j=1}^N l_{ij}x_j(t) + \gamma \sum_{j=1}^N l_{ij}v_j(t) \end{cases} \quad (37)$$

where $x_i(t)$ is the location of node i and $v_i(t)$ is its moving speed.

In [14], the following model of second order consensus with decay was proposed

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \alpha \sum_{j=1}^N l_{ij}x_j(t) + \beta \sum_{j=1}^N l_{ij}v_j(t) \\ \quad - \eta v_i(t) - \gamma x_i(t) \end{cases} \quad (38)$$

It is clear that the model (38) is a generalization of the model (37). Here, we first discuss the model (38), which can be written as

$$\frac{dZ(t)}{dt} = Z(t)\tilde{L}_1^\top \quad (39)$$

where $X(t) = [x^1(t), x^2(t), \dots, x^N(t)] \in R^{n,N}$, $V(t) = [v^1(t), v^2(t), \dots, v^N(t)] \in R^{n,N}$, $Z(t) = [X(t), V(t)]$, $L = [l_{ij}]$, and

$$\tilde{L}_1 = \begin{bmatrix} 0 & I_N \\ \alpha L - \gamma I_N & \beta L - \eta I_N \end{bmatrix}$$

Here, we give some geometric structure of the coupling matrix \tilde{L}_1 .

It is clear that any eigenvector of \tilde{L}_1 with the eigenvalue λ_i should be of the form $\bar{W}^i = [\bar{W}_1^i, \bar{W}_2^i]^\top$, where $\bar{W}_1^i = [w_1^i, \dots, w_N^i]^\top$, and $\bar{W}_2^i = \lambda_i \bar{W}_1^i = [\lambda_i w_1^i, \dots, \lambda_i w_N^i]^\top$. Moreover, $(\alpha L - \gamma I_N)\bar{W}_1^i + \lambda_i(\beta L - \eta I_N)\bar{W}_1^i = \lambda_i^2 \bar{W}_1^i$.

Therefore, we have following Lemmas, which reveal the relations between the eigenvalues and eigenvectors of L and \tilde{L}_1 .

Lemma 2. Suppose that $W^i = [w_1^i, \dots, w_N^i]^\top$ is the right eigenvector of L corresponding to eigenvalue μ_i , then the vector $\bar{W}^i = [w_1^i, \dots, w_N^i, \lambda_i w_1^i, \dots, \lambda_i w_N^i]^\top$ is the eigenvector of \tilde{L}_1 corresponding to eigenvalue λ_i , where λ_i satisfies $(\alpha\mu_i - \gamma) + (\beta\mu_i - \eta)\lambda_i = \lambda_i^2$.

In the following, we will discuss following four cases separately.

Case 1. $\gamma = 0, \eta = 0$.

Case 2. $\gamma > 0, \eta > 0$.

Case 3. $\gamma = 0, \eta > 0$.

Case 4. $\gamma > 0, \eta = 0$.

Case 1. $\gamma = 0, \eta = 0$. In this case, we have $\alpha\mu_i + \beta\mu_i\lambda_i = \lambda_i^2$.

Lemma 3. In case $\gamma = 0, \eta = 0$, "0" is an eigenvalue of \tilde{L}_1 with multiplicity two. Furthermore, if $\xi = [\xi_1^i, \dots, \xi_N^i]^\top$ is the left eigenvector of L corresponding to eigenvalue "0", then $\bar{\xi} = [0, \dots, 0, \xi_1^i, \dots, \xi_N^i]^\top$ and $\bar{\bar{\xi}} = [\xi_1^i, \dots, \xi_N^i, 0, \dots, 0]^\top$ are two left root vectors of \tilde{L}_1 corresponding to eigenvalue "0". That is

$$\bar{\xi}^\top \tilde{L}_1 = \mathbf{0}_{2N}$$

$$\bar{\bar{\xi}}^\top \tilde{L}_1 = \bar{\bar{\xi}}$$

Moreover, $\bar{\mathbf{1}} = [0, \dots, 0, 1, \dots, 1]^\top$ and $\bar{\bar{\mathbf{1}}} = [1^\top, 0^\top]^\top = [1, \dots, 1, 0, \dots, 0]^\top$ are two right root vectors of \tilde{L}_1 corresponding to eigenvalue "0".

$$\tilde{L}_1 \bar{\mathbf{1}} = \mathbf{0}_{2N}$$

$$\tilde{L}_1 \bar{\bar{\mathbf{1}}}^\top = \bar{\bar{\mathbf{1}}}$$

Let $\tilde{L}_1^\top = S_1 J^* S_1^{-1}$ be the Jordan decomposition of L_1^\top , where $J^* = \text{diag}\{J_1, \dots, J_l\}$ is a block diagonal matrix corresponding to different eigenvalues λ_k , $k = 1, \dots, l$, with $\lambda_1 = 0$ and

$$J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

It can be seen that the first column of S_1 is $\bar{\xi}$ and second is $\bar{\bar{\xi}}$. The first row of S_1^{-1} is $\bar{\mathbf{1}}^\top$ and second is $\bar{\bar{\mathbf{1}}}^\top$.

Denoting $\bar{Z}(t) = Z(t)S_1$, we have

$$\frac{d\bar{Z}(t)}{dt} = \bar{Z}(t)J^* \quad (40)$$

where $\bar{Z}(t) = [\bar{Z}_1(t), \dots, \bar{Z}_l(t)]$, $\bar{Z}_k(t) = [\bar{Z}_{k_1}(t), \dots, \bar{Z}_{k_{m_k}}(t)]$.

And for $1 \leq k \leq l$, we have

$$\frac{d\bar{Z}_{k_1}(t)}{dt} = \lambda_k \bar{Z}_{k_1}(t) \quad (41)$$

$$\begin{aligned} \frac{d\bar{Z}_{k_{p+1}}(t)}{dt} &= \lambda_k \bar{Z}_{k_{p+1}}(t) + \bar{Z}_{k_p}(t), \\ 1 \leq p \leq m_k - 1 \end{aligned} \quad (42)$$

where λ_k is an eigenvalue of the coupling matrix L_1 , which may be a complex number.

Therefore, in case the real part of $\lambda_k < 0$, $k = 2, \dots, l$, then $\bar{Z}_p(t) = O(e^{-\alpha t})$, $1 \leq p \leq m_k - 1$, $k = 2, \dots, l$.

As for $k = 1$, we have

$$\frac{d\bar{Z}_{1_1}(t)}{dt} = 0 \quad (43)$$

$$\frac{d\bar{Z}_{1_2}(t)}{dt} = \bar{Z}_{1_1}(t) \quad (44)$$

which imply

$$\bar{Z}_{1_1}(t) = \bar{Z}_{1_1}(0) = Z(0)\bar{\xi} = V(0)\xi$$

$$\bar{Z}_{1_2}(t) = \bar{Z}_{1_2}(0) + t\bar{Z}_{1_1}(0) = X(0)\xi + tV(0)\xi$$

Noticing first row of S_1^{-1} is $\bar{\mathbf{1}}^\top$ and second is $\bar{\bar{\mathbf{1}}}^\top$, we have

$$V(t) = V(0)\xi + O(e^{-\alpha t})$$

$$X(t) = X(0)\xi + tV(0)\xi + O(e^{-\alpha t})$$

which means

$$\begin{cases} x_i(t) = \sum_{j=1}^N \xi_i(x_j(0) + tv_j(0)) + O(e^{-\alpha t}) \\ v_i(t) = \sum_{i=1}^N \xi_i v_i(0) + O(e^{-\alpha t}) \end{cases} \quad (45)$$

Remark 7. The approach of analysis for second order consensus of the model (15) can also be traced to the papers [3], [4]. The only difference lies in that

$\gamma \neq 0$ or $\eta \neq 0$.

In this case, let $\tilde{L}_1^\top = \tilde{S}_1 \tilde{J}^* \tilde{S}_1^{-1}$ be the Jordan decomposition of L_1^\top , where $\tilde{J}^* = \text{diag}\{\tilde{J}_1, \dots, \tilde{J}_l\}$ is a block diagonal matrix corresponding to different eigenvalues λ_k , $k = 1, \dots, l$, with λ_1, λ_2 satisfy $\lambda^2 + \eta\lambda + \gamma = 0$.

Denoting $\bar{Z}(t) = Z(t)S_1$, we have $\bar{Z}(t) = [\bar{Z}_1(t), \dots, \bar{Z}_l(t)]$, $\bar{Z}_k(t) = [\bar{Z}_{k_1}(t), \dots, \bar{Z}_{k_{m_k}}(t)]$.

and for $3 \leq k \leq l$, we have

$$\frac{d\bar{Z}_{k_1}(t)}{dt} = \lambda_k \bar{Z}_{k_1}(t) \quad (46)$$

$$\begin{aligned} \frac{d\bar{Z}_{k_{p+1}}(t)}{dt} &= \lambda_k \bar{Z}_{k_{p+1}}(t) + \bar{Z}_{k_p}(t), \\ 1 \leq p \leq m_k - 1 \end{aligned} \quad (47)$$

where λ_k is an eigenvalue of the coupling matrix \tilde{L}_1 , which may be a complex number.

If the real part of $\lambda_k < 0$, $k = 3 \cdots l$. And denote

$$\alpha = \max_{k=2, \dots, p} \operatorname{Re}(\lambda_k)$$

then $\bar{Z}_p(t) = O(e^{-\alpha t})$, $1 \leq p \leq m_k - 1$, $k = 2 \cdots l$.

Case 2. $\gamma > 0$, $\eta > 0$. Direct verification gives $\operatorname{Re}(\lambda_1) < 0$, $\operatorname{Re}(\lambda_2) < 0$. Then

$$V(t) = O(e^{-\alpha t})$$

$$X(t) = O(e^{-\alpha t})$$

Case 3. $\gamma = 0$, $\eta > 0$. Then, $\lambda_1 = 0$ with left eigenvector $[\xi, \frac{1}{\eta}\xi]$ and right eigenvector $[1, 0]^T$ $\lambda_2 = -\eta < 0$. Then,

$$X(t) = \xi^T X(0) + \frac{1}{\eta} \xi^T V(0) + O(e^{-\alpha t})$$

$$V(t) = O(e^{-\alpha t})$$

Case 4. $\gamma > 0$, $\eta = 0$. Then, $\lambda_1 = i\gamma^{1/2}$ and $\lambda_2 = -i\gamma^{1/2}$. Corresponding left eigenvectors are $[\gamma^{1/2}\xi^T, i\xi^T]$ and $[\gamma^{1/2}\xi^T, -i\xi^T]$, respectively. Corresponding right eigenvectors are $[1_N^T, i\gamma^{1/2}\mathbf{1}_N^T]^T$ and $[i\gamma^{1/2}\mathbf{1}_N^T, -1_N^T]^T$, respectively. The consensus trajectories are periodic.

The remaining work is to compute λ_i by solving the equation

$$(\alpha\mu_i - \gamma) + (\beta\mu_i - \eta)\lambda_i = \lambda_i^2$$

Remark 8. It should be pointed out that in previous models (37) and (38), the coupling terms $\sum_{j=1}^N l_{ij}x_j(t) - \gamma x_i(t)$ is added to $\dot{v}_i(t)$. This is quite strange.

Intuitively, it is questionable by adding to add coupling terms $\sum_{j=1}^N l_{ij}x_j(t) - \gamma x_i(t)$ is added to $\dot{v}_i(t)$ to make $v_i(t)$ reach consensus.

Mathematically, this coupling term does not make contributions to realization of consensus. On the contrary, it brings more complicated dynamical behaviors. For example, to make model (37) reach consensus, in [11], the parameter γ should be large enough. To make model (38) reach consensus, in [14], the parameter α should be small enough. These two conditions confirm that the negative effects brought by the coupling terms $\sum_{j=1}^N l_{ij}x_j(t) - \gamma x_i(t)$

Therefore, these two models are not proper in both theoretical respect and practice.

In reality, the following model is reliable and easy to use.

$$\begin{cases} \dot{x}_i(t) = v_i(t) + \sum_{j=1}^N (l_{ij} - \gamma)x_j(t) \\ \dot{v}_i(t) = \sum_{j=1}^N (l_{ij} - \eta)v_j(t) \end{cases} \quad (48)$$

As done in previous section, it is easy to prove that for model (48), we have

$$\begin{cases} x_i(t) = \sum_{i=1}^N \xi_i(x_j(0) + \gamma + t(v_j(0) + \eta)) + O(e^{-\alpha t}) \\ v_i(t) = \sum_{i=1}^N \xi_i v_i(0) + \eta + O(e^{-\alpha t}) \end{cases} \quad (49)$$

Conclusions

In this paper, we revisit the topic "synchronization and consensus in a unified viewpoint". Several protocols on this topic are also revisited and the relationships between them

are addressed. It is pointed out that the model introduced in [2], [4] and the approach provided there is universal. Many existed synchronization and consensus models and their stability behavior analysis can be derived easily from the theoretical results given in [2], [4]. These models include consensus and synchronization of linear coupled nonlinear (or linear) systems, observed-based linear systems, second order consensus with decays (or without decays) and many others.

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